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Causal structures and holography

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Abstract: We explore the description of bulk causal structure in a dual field theory. We observe that in the spacetime dual to a spacelike non-commutative field theory, the causal structure in the boundary directions is modified asymptotically. We propose that this modification is described in the dual theory by a modification of the micro-causal light cone. Previous studies of this micro-causal light cone for spacelike non-commutativite field theories agree with the expectations from the bulk spacetime. We describe the spacetime dual to field theories with lightlike non-commutativity, and show that they generically have a drastic modification of the light cone in the bulk: the spacetime is non-distinguishing. This means that the spacetime while being devoid of closed timelike or null curves, has causal curves that are "almost closed". We go on to show that the micro-causal light cone in the field theory agrees with this prediction from the bulk.

Keywords: AdS-CFT and dS-CFT Correspondence, Penrose limit and pp-wave background, Non-Commutative Geometry.

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Contents

1. Introduction

The advent of the AdS/CFT correspondence [1] has revolutionised our understanding of quantum gravity, and has led to important insights into both gravity and strongly coupled gauge dynamics. However, a considerable number of important conceptual questions remain open, especially those pertaining to a detailed understanding of gravitational dynamics. In particular, we still lack full understanding of how field theory on a fixed background can be approximated by a local theory with dynamical metric in the bulk.

One major issue is that the bulk spacetime has some dynamically determined causal structure, whereas the dual field theory lives in flat space, with the usual fixed light cone. The description of the bulk causal structure from the dual point of view has been the subject of a number of investigations in the AdS/CFT correspondence. Bulk causality in global AdS spacetime is consistent with boundary causality [2]; for example, propagation through the bulk is never faster than propagation along the boundary (this was extended to asymptotically AdS spacetimes satisfying the weak energy condition in [3]). When we restrict to Poincaré-invariant states, the light cone in the boundary directions will automatically agree in the bulk and boundary, and the focus is on understanding the causal structure in the bulk radial direction. In [4], Kabat and Lifschytz proposed a scale/radius description, where causality in the radial direction is enforced by a speed limit on varying scale size in the field theory. If we violate Poincaré invariance, for example by considering a black hole spacetime in the bulk (corresponding to finite temperature in the field theory), the bulk causal structure will not in general agree with the field theory one even in the boundary directions. The extension of Kabat & Lifschytz's analysis to such cases was considered in [5]. However, for asymptotically AdS spacetimes, the causal structure changes only in the interior of the spacetime, which makes it difficult to pose sharp questions about the interpretation of these changes in the field theory.

In this paper, we will consider extreme examples of changes in the bulk causal structure: spacetimes dual to non-commutative field theories. Since the non-commutativity changes the structure of the field theory even in the ultraviolet, the bulk spacetime is no longer asymptotically AdS. The point of interest to us is that the causal structure of the bulk spacetime is drastically modified: in the limit as we approach the boundary, the light cone in the boundary directions ceases to depend on the directions in which non-commutativity is turned on. Thus, propagation through the bulk can take us outside the flat space light cone in these directions. We want to understand how this is described from the field theory point of view. One would expect that the non-locality of the non-commutative field theory plays a crucial role, and we will argue that this is indeed the case. In [6, 7], it was argued that the non-locality leads to modifications of the micro-causal light cone: in particular, it does not agree with the naive causal structure of the background flat space the field theory is defined in. We will show in a simple toy example, scalar non-commutative field theory, that these modifications agree with the light cone in the boundary directions of the bulk spacetime. Extending this result to non-commutative gauge theories is somewhat tricky owing to issues related to absence of local gauge invariant observables.

Nevertheless, we will argue based on the known properties of correlation functions of gauge invariant operators, that we indeed expect to see a modified causal structure in the field theory.

We study the modifications of the causal structure for both the spacelike non-commutativity, previously studied in [6, 7], and for lightlike non-commutativity. The lightlike case is especially interesting because the corresponding spacetime geometry has a very radical modification of its causal structure. The spacetime becomes *non-distinguishing*, meaning that distinct points of the spacetime have the same causal future and past. Such a spacetime 'almost has' closed timelike curves (CTCs); it is in the borderline area between being clearly causally well-behaved and clearly pathological. Since the discovery of supersymmetric spacetimes with CTCs [8], there has been a lot of discussion of whether spacetimes with CTCs are admissible backgrounds in string theory. Most of the discussion has centered on some string version of chronology protection; for example [9] related chronology violation in AdS/CFT to unitarity loss, while [10] argue for a holographic chronology protection mechanism. In light of the latter proposal that holography could 'cut off' the region of CTCs, it is remarkable to observe that lightlike non-commutativity provides an example where a non-distinguishing spacetime has a well-behaved holographic dual description. We will suggest that the non-distinguishing character of this spacetime is reflected in the micro-causal structure of the lightlike non-commutative field theory. Understanding this case may also be helpful for understanding the dual description of plane wave spacetimes, as they are also on the borderline (although they are distinguishing, they are not globally hyperbolic, and do not have Cauchy surfaces). Non-distinguishing examples of pp-waves have also recently been constructed [11 − 13].

In the next section, we will review the construction of the dual spacetime for a spacelike non-commutative deformation of $\mathcal{N} = 4$ SYM [14, 15], and discuss the features of the bulk light cone. In section 3, we consider the micro-causality in spacelike non-commutative field theory. We review the argument of [6, 7] that the micro-causality condition for spacelike non-commutative field theories is to be imposed inside a 'light wedge'. In [6] it was argued that for a non-commutative field theory on $\mathbf{R}^{3,1}$ with $\theta^{23} \neq 0$, the breaking of Lorentz symmetry by non-commutativity would enlarge the region where the commutator of fields is non-vanishing from the usual light cone to a light wedge respecting the unbroken $SO(1,1) \times SO(2)$ symmetry. As discussed in [7], the non-local character of the field theory plays an essential role in this enlargement. We show that this light wedge agrees with the bulk results and suggest a framework for exploring these issues in non-commutative gauge theories.

We then turn to the consideration of lightlike non-commutativity. In section 4 we construct the bulk spacetime by applying the "Null Melvin Twist", a solution-generating technique discussed in [16, 17]. We get our solution by applying this twist to the extremal D3-brane, and then taking a decoupling limit. We demonstrate that the spacetime is non-distinguishing using the arguments of [11, 12], and briefly sketched in the appendix. We then show in section 5 that this spacetime is the holographic dual description of a theory with lightlike non-commutativity. (This solution and its dual description were previously obtained in [18].) It was shown in [19] that theories with lightlike non-commutativity are well-defined quantum field theories, albeit with non-local interactions. We thus have

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an example where a non-distinguishing spacetime has a non-perturbative quantum description. We go on to show that any spacetime dual to a generic theory with lightlike non-commutativity will be non-distinguishing. Extending our arguments from the spacelike case, we are able to demonstrate that the micro-causal structure in a simple scalar field theory with lightlike non-commutativity reproduces the predicted causal structure inferred from these non-distinguishing spacetimes. Commutators of local operators can be shown to vanish only at equal light cone time, respecting the unbroken galilean subgroup of the Lorentz group (in the presence of lightlike non-commutativity). We further argue that subtleties relating to gauge invariance will not change the result significantly in the case of lightlike non-commutative gauge theories and conclude with a brief discussion. Some details of the field theory calculations and more general solutions obtained by the null Melvin twist are described in the appendices. A short summary of the essential physical ideas can be found in [20].

2. Spacelike non-commutativity: bulk light cone

We begin by discussing the situation in the case of spacelike non-commutative field theories. In this section, we explore the causal properties of the dual spacetime, and note that it has a deformed light cone even in the asymptotic region. We consider geodesics which remain at large radius to study the features of this light cone.

The string frame metric for the spacetime dual to spacelike non-commutative $\mathcal{N} = 4$ SYM is [14, 15]

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2\right) + \frac{R^2\, r^2}{R^4 + \gamma^2\, r^4}\left(dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2}\, dr^2 + R^2\, d\Omega_5^2\,. \qquad (2.1)$$

There are non-trivial 3-form and 5-form fluxes in the above background and a varying dilaton, but these are not going to be relevant to our present discussion. R as usual denotes the AdS radius and γ is related to the non-commutative parameter in the field theory as $\theta^{23} = \gamma$.

This dual geometry is obtained by thinking of the spacelike non-commutative Yang-Mills theory as arising as the low-energy limit of the theory on D3-branes in a constant background B-field. The supergravity configuration sourced by these D3-branes can be obtained by introducing the B-field through a twisting operation. To twist, start from the D3-branes in empty space, T-dualize to D2-branes, and then consider instead the compactification of the D2-brane solution on a tilted torus. T-dualizing along this tilted direction will give D3-branes in a B-field background. Applying this twisting operation to the D3-brane metric, and then taking the decoupling limit to focus on the near-horizon geometry, gives the spacetime (2.1).

While the geometry (2.1) is close to $\mathrm{AdS}_5 \times \mathbf{S}^5$ for small r, it has vastly different asymptotics. In particular, $g_{x_i x_i} \to 0$ as $r \to \infty$ for $i = 2, 3$, which will give rise to the deformation of the light cone we are interested in. This makes it hard to define a boundary for the spacetime, but we will show later that the deformation of the light cone is nonetheless reflected in field theory micro-causality behaviour.

Suppose we look at the metric induced on a fixed $r = r_0$ surface with $r_0 \gg R$, i.e., we think of the bulk defining an UV regulated version of the non-commutative field theory. In this case, the induced metric on the surface (ignoring the $\mathbf{S}^5$ part) reads:

$$ds^2 \approx \frac{r_0^2}{R^2} \left(-dt^2 + dx_1^2 + \frac{R^4}{\gamma^2 \, r_0^4} \, \left(dx_2^2 + dx_3^2\right) \right) . \tag{2.2}$$

At any fixed value of r_0, we can rescale the coordinates (t, x_i) to convert (2.2) into a flat metric on $\mathbf{R}^{3,1}$. However, the scaling is non-homogeneous, and in particular, as we remove the cutoff we will have to scale the shrinking directions (x_2, x_3) by a diverging amount, whereas we would expect the field theory background to be defined by a homogeneous rescaling of coordinates, as in the usual commutative case. This would lead us to conclude that the relevant light cone for the field theory is independent of the non-commuting directions $\{x_2, x_3\}$. Alternately, as is apparent from the full metric (2.1), the metric factor in front of the non-commuting directions falls off faster than that in front of the $\mathbf{S}^5$ part, and therefore should not contribute to the light cone asymptotically.

We would like to argue that this change in the bulk light cone is reflected in observable quantities in the dual field theory. One approach would be to study the two-point function in the boundary, which should be determined, by the usual logic, by the bulk-to-bulk propagator for the appropriate supergravity field in the background (2.1) in the limit $r \to \infty$. By examining the behaviour of the bulk Greens functions we should be able to extract the detailed properties of the bulk light cone, and in particular its asymptotic behaviour. However, the Klein-Gordon equation is rather formidable (it is related to the Mathieu equation [15]), and so explicit determination of the propagator is difficult. Nevertheless, we can approximate it by studying the geodesics in the bulk spacetime. We will now show that the bulk light cone indeed degenerates into a light wedge asymptotically in the geodesic approximation.

Ignoring motion on the $\mathbf{S}^5$ part of the geometry and setting $R = 1$ for simplicity, we have the equation for timelike geodesics

$$-1 = r^2 \, \left(-\dot{t}^2 + \dot{x}_1^2\right) + \frac{r^2}{1 + \gamma^2 \, r^4} \, \left(\dot{x}_2^2 + \dot{x}_3^2\right) + \frac{\dot{r}^2}{r^2} \, , \tag{2.3}$$

where $\dot{} \equiv \frac{d}{d\tau}$, with τ the affine parameter along the geodesics. The Killing symmetries $\left(\frac{\partial}{\partial t}\right)^a$ and $\left(\frac{\partial}{\partial x^i}\right)^a$ determine the conserved energy and momenta (p_t, p_i), respectively. The geodesic motion reduces to a classical problem for the zero energy trajectories of a particle in an effective potential $V_{\text{eff}}(r)$:

$$\begin{aligned} \dot{r}^2 + V_{\text{eff}}(r) &= 0 \, , \\ \dot{t} = \frac{p_t}{r^2} \, , \qquad \dot{x}_1 &= \frac{p_1}{r^2} \, , \qquad \dot{x}_i = \frac{1 + \gamma^2 \, r^4}{r^2} \, p_i \, , \\ V_{\text{eff}}(r) &= r^2 + \left[-p_t^2 + p_1^2 + \left(1 + \gamma^2 \, r^4\right) \left(p_2^2 + p_3^2\right)\right] . \end{aligned} \tag{2.4}$$

Let us consider geodesics that travel an infinitesimal distance in the radial direction. Choose $r = r_0 \, (1 - \varepsilon)$ with $\varepsilon \ll 1$ such that $V_{\text{eff}}(r_0) = 0$. Then we have

$$r_0^2 \, \dot{\varepsilon}^2 \approx 2 \, r_0^2 \, \varepsilon \left(1 + 2 \, \gamma^2 \, r_0^2 \, \left(p_2^2 + p_3^2\right)\right) \equiv 2 \, \varepsilon \, \alpha \, . \tag{2.5}$$

This allows us to approximately integrate the geodesic equations to obtain, for instance,

$$\Delta t = \int d\tau \, \frac{p_t}{r^2} \approx \frac{p_t}{r_0^2} \int d\tau = \frac{p_t}{r_0 \sqrt{\alpha}} \sqrt{2\,\varepsilon} \tag{2.6}$$

and

$$\Delta x_2 \approx \frac{p_2}{r_0 \sqrt{\alpha}} \left(1 + \gamma^2 \, r_0^4\right) \sqrt{2\,\varepsilon} \,. \tag{2.7}$$

One can clearly make the distance travelled by the geodesics in individual directions large, by choosing appropriate values for the momenta. However, the motion is confined to the region in the immediate vicinity of the classical bulk light cone, as the proper interval remains small:

$$\begin{aligned} -\,\Delta t^2 + \Delta x_1^2 + \frac{\Delta x_2^2 + \Delta x_3^2}{1 + \gamma^2 \, r_0^4} &\approx \frac{2\,\varepsilon}{\alpha \, r_0^2} (-p_t^2 + p_1^2 + (1 + \gamma^2 \, r_0^4)(p_2^2 + p_3^2)) \\ &\approx -\frac{2\,\varepsilon}{\alpha} \,, \end{aligned} \tag{2.8}$$

where we made use of the fact that $V_{\text{eff}}(r_0) = 0$.

Thus, bulk geodesics can relate points inside the light cone given by

$$-\Delta t^2 + \Delta x_1^2 + \frac{\Delta x_2^2 + \Delta x_3^2}{1 + \gamma^2 \, r_0^4} = 0 \,, \tag{2.9}$$

and as we remove the cutoff by taking $r_0 \to \infty$, the light cone in the boundary directions will approach a 'light wedge':

$$-\Delta t^2 + \Delta x_1^2 = 0 \,. \tag{2.10}$$

This is the advertised modification of the bulk causal structure. Although we have restricted our attention to bulk geodesics, we will have essentially obtained the same answer by looking at say the free scalar propagator for (2.1). Having obtained a prediction from the bulk perspective, we now turn to looking for signs of this change in the causal structure in the field theory.

3. Micro-causality in non-commutative field theories

The field theory dual to the geometry (2.1) is $\mathcal{N} = 4$ SYM theory in flat four-dimensional spacetime, with a constant non-commutativity parameter $\theta^{23} = \gamma$. We want to show that the above 'light wedge' structure is also naturally reflected in this field theory. As we are dealing with a Lorentz non-invariant non-local field theory, it is not *a priori* obvious what the causal relations in the field theory are. We will see that if we define the light cone of the field theory to be the micro-causal light cone, that is, the boundary of the region where the commutator of elementary fields is non-vanishing, then it will agree with the light wedge predicted from the bulk.

3.1 Non-commutative scalar field dynamics: a toy model

The micro-causal light cone for spacelike non-commutative field theories was studied in [6], and further clarified recently in [7]. In [6], a heuristic argument was given for a modification of the micro-causality condition. The non-commutativity in a spatial $\mathbf{R}^2 \subset \mathbf{R}^{3,1}$, $\theta^{23} \neq 0$, breaks Lorentz invariance in the field theory from $\mathrm{SO}(3,1) \to \mathrm{SO}(1,1) \times \mathrm{SO}(2)$. As a result it was suggested that one should not demand causal behaviour with respect to the full $\mathrm{SO}(3,1)$ invariance, but rather only with respect to the smaller symmetry $\mathrm{SO}(1,1)$. This corresponds to the invariance of the light wedge obtained above (although the reduced symmetry is not sufficient to fix the form of the light cone).

That is, the authors of [6] propose that instead of requiring that the fields (labeled collectively as $\Phi(x)$) commute (or anti-commute for fermions) across spacelike separated points,

$$\langle\, [\Phi(x), \Phi(y)]_\pm \,\rangle = 0 \qquad \text{for } (x-y)^2 > 0\,. \tag{3.1}$$

the appropriate micro-causality condition for non-commutative field theories would be to only impose

$$\langle\, [\Phi(x), \Phi(y)]_\pm \,\rangle = 0 \qquad \text{for } (x-y)_c^2 > 0\,. \tag{3.2}$$

where $(x-y)_c^2$ is the separation in the commuting directions,

$$(x-y)_c^2 = -(x^0-y^0)^2 + \sum_{i\,\in C} (x^i - y^i)^2\,, \qquad \text{where } C = \{i : \theta^{ik} = 0\ \forall\, k\}\,. \tag{3.3}$$

For example with $\theta^{23} \neq 0$ we have $(x-y)_c^2 = -(x^0-y^0)^2 + (x^1-y^1)^2$.

This argument has two weaknesses: since it appeals to the violation of Lorentz invariance, it would appear to apply whenever we have Lorentz violating interactions, and not only in non-commutative field theories. Also, the $\mathrm{SO}(1,1)$ symmetry does not really determine a light cone; while the light wedge used in (3.2) respects $\mathrm{SO}(1,1)$, so does the original light cone of (3.1). In [7], these weaknesses were addressed by performing an explicit perturbative calculation of the commutator for non-commutative field theory, showing that the micro-causality condition (3.2) is correct for spacelike non-commutative field theories, whereas (3.1) remains correct for theories with local Lorentz violating interactions. This shows that the non-local character of the non-commutative interactions is important.

We will give a brief derivation of this result [7]. The authors consider spacelike non-commutative φ^3 theory in $\mathbf{R}^{5,1}$ for simplicity,

$$S = \int d^6x \left(-\frac{1}{2}\,\partial_\mu\varphi \star \partial^\mu\varphi - \frac{1}{2}\, m^2\, \varphi\star\varphi + \frac{1}{3!}\, g\,\varphi\star\varphi\star\varphi \right). \tag{3.4}$$

The strategy is to calculate the expectation value of the Heisenberg picture field operator between two states $|\,\alpha\,\rangle$ and $|\,\beta\,\rangle$, obtained by time-evolution of the perturbative vacuum $|\,0\,\rangle$,

$$\mathcal{M} = \langle\, \alpha |\; [\varphi_H(x_1), \varphi_H(x_2)]\; |\, \beta\,\rangle\,. \tag{3.5}$$

Given a perturbation expansion of

$$\mathcal{M} = \sum_n g^n \, \mathcal{M}^{(n)} \,, \tag{3.6}$$

one can study $\mathcal{M}^{(n)}(x)$ to determine the domains where it is guaranteed to vanish. We have used translational invariance to write $x = x_1 - x_2$. This will provide us with the definition of the micro-causal light cone in field theory.

The calculation of $\mathcal{M}^{(n)}(x)$ proceeds in the standard fashion and some of the details can be found in appendix A. The main point is that we can write $\mathcal{M}^{(n)}(x)$ in an integral representation as

$$\mathcal{M}^{(n)}(x) = N_n \left[\int \prod_i d\ell_i \, G_n(\ell_i, x) - \int \prod_i d\ell_i \, G_n(\ell_i, -x) \right] , \tag{3.7}$$

where the integrand $G_n(\ell_i, x)$ generically takes the form

$$G_n(\ell_i, x) = \exp\left(i \, \frac{x^2}{2u_x} \sum_i \ell_i - i \, \frac{m^2 u_x}{2} \sum_i \frac{1}{\ell_i} \right) H_n(\ell_i, x) \,. \tag{3.8}$$

The integration variables ℓ_i are the independent light cone momenta after imposition of momentum conservation and N_n are constants. The details of the interaction are contained in the kernel $H_n(\ell_i, x)$. In writing (3.7), we have rewritten various amplitudes in terms of light cone variables for ease of computation: $x^2 = -2 \, u_x \, v_x + \vec{x}^2$.

The behaviour of the integrand $G_n(\ell_i, x)$ in the complex space parameterised by ℓ_i is crucial in determining the convergence properties of the integral. For instance, at tree-level in perturbation theory we have

$$G_0(\ell, x) = \frac{1}{4 \pi \ell} \left(\frac{i \, \ell}{2 \pi \, x} \right)^2 \exp\left(i \, \ell \frac{x^2}{2 \, u_x} - i \, \frac{m^2 \, u_x}{2 \, \ell} \right) \tag{3.9}$$

Given this, the usual contour deformation arguments can be used to show that

$$\mathcal{M}^{(0)}(x) = 0 \quad \Longleftrightarrow \quad x^2 = (x_1 - x_2)^2 > 0 \,, \tag{3.10}$$

which is the expected result for the free field theory (recall that the non-commutative deformation affects quadratic terms only by the presence of irrelevant phase factors). To be precise, assuming $u_x > 0$ without loss of generality, for $x^2 > 0$ we can rotate the contour of integration for the ℓ integral as $\ell \to i \, \ell$ in integrating $G_0(\ell, x)$ (and $\ell \to -i \, \ell$ for $G_0(\ell, -x)$). The two integrals then are convergent and moreover the finite answers cancel in the difference; hence the commutator vanishes. For timelike separation $x^2 < 0$, there is no deformation of the contour that gives a convergent integral; hence generically the difference of the two integrals is non-vanishing and we are led to (3.10).

For non-commutative field theories, we intuitively expect to see the first sign of interesting effects at one-loop, i.e., in $\mathcal{M}^{(2)}$, for we have qualitatively new feature in perturbation theory in the form of non-planar Feynman graphs. We should expect that the phase factors involved in the non-commutative φ^3 interaction modify the properties of $G_2(\ell_i, x)$ and thereby deform the nature of the light cone. Detailed calculations [7] show that this expectation is indeed borne out. We in fact find that (see appendix A for details)

$$G_2(\ell_1, \ell_2, x) \sim \exp\left(i\,(\ell_1 + \ell_2)\frac{x_c^2}{2\,u_x}\right), \tag{3.11}$$

implying that

$$\mathcal{M}^{(2)}(x) = 0 \quad \Longleftrightarrow \quad x_c^2 = (x_1 - x_2)_c^2 > 0\,, \tag{3.12}$$

with x_c^2 is as defined in (3.3). This follows by repeating the argument for the convergence of the integrals with the modified integrand (3.11). So we see that the micro-causality condition in spacelike non-commutative field theories is modified at one-loop. In particular, note that the presence of the non-commutative interactions demand that the expectation value of the field commutator vanish only outside a light-wedge as surmised earlier.[1]

This micro-causal light wedge agrees with the prediction of the spacetime in the previous section. This agreement should be considered as qualitative evidence that the micro-causal structure in the field theory is indeed related to the light cone determined by the bulk spacetime.

3.2 Micro-causality in non-commutative gauge theories

We have shown that the micro-causality in a scalar non-commutative field theory agrees with the bulk light cone. However, this scalar field theory is only a toy model, and there are important differences which make it unclear if this behaviour will generalize to the non-commutative gauge theory which is actually dual to the bulk spacetime (2.1). We will now review the salient differences, and argue that we nevertheless still expect to see a reflection of the bulk light cone in the micro-causality of the non-commutative gauge theory.

The first difference has to do with the structure of UV divergences in gauge theories. As remarked in the footnote above, the modification of the micro-causality condition is related to the IR divergences seen in perturbation theory for non-commutative dynamics. Although explicit calculation confirms the presence of such IR poles in a general non-commutative gauge theory [22], they will be absent for the $\mathcal{N} = 4$ non-commutative SYM theory we're interested in because of the supersymmetry. Thus, the modification of the micro-causality condition for gauge theories can't arise in the same way as it did in the scalar theory studied above.

There is however another important difference, which we will argue can lead to a similar modification of the micro-causality condition by a more subtle route. In non-commutative gauge theories there are no gauge invariant local operators (essentially because translation in the non-commutative directions is equivalent to a gauge transformation), so it is unclear why we should even consider the micro-causal structure based on local fields. An over-complete set of non-local gauge invariant observables was constructed in [23 − 25]. The

[1] As an interesting aside, it should be possible to show the modification of the micro-causality condition using the 1PI effective action derived for non-commutative ϕ^3 theory in [21]. The inverse propagator (in momentum space) for this theory is $p^2 + m^2 + g^2\,h/p \circ p$, with $p \circ q = p_\mu\,(\theta^2)^{\mu\nu}\,q_\nu$. This propagator has extra poles in the complex momentum plane at $p \circ p = -h\,g^2/(p_c^2 + m^2)$, where p_c is the restriction of the momentum to the commutative sub-space. The presence of these poles induces a delta function for the momentum along the non-commutative directions. This ultra localization in momentum space is the origin of the light-wedge (3.3).

idea was to string a local operator such as $\mathrm{Tr}(F_{\mu\nu}\, F^{\mu\nu})$ (which is of course gauge invariant in the commutative limit) with an open Wilson line and take its Fourier transform. This defines a local gauge invariant operator in momentum space $\mathcal{O}(k)$. They calculated correlation functions of the operators $\mathcal{O}(k)$ and showed that the correlation functions grow exponentially in momenta

$$\langle \mathcal{O}(k)\, \mathcal{O}(-k) \rangle \sim \exp\left(\sqrt{\frac{g_{YM}^2\, N}{4\,\pi}\, |k\,\theta|\, |k|} \right). \tag{3.13}$$

This exponential growth of correlation functions may also be seen from the supergravity dual [24]. In writing the above we have dropped some regularization dependent terms which will be unimportant for properly renormalised correlators. We want to suggest that the exponential growth of the correlation functions signals a modification of the micro-causality conditions for these operators.

The clue comes from studies of a similar behaviour in another non-local quantum field theory discovered in the past decade, little string theory, which is a Poincaré invariant theory in six dimensions with a mass scale l_s^{-1} , living on the world-volume of NS5-branes. These theories admit gauge invariant local operators in momentum space and their correlation functions also grow exponentially in momenta [26]. This fact was used in [27] to argue that little string theories are quasi-local theories, and that a modified notion of micro-causality could be defined for such theories.

In local quantum field theories, correlation functions in position space, the Wightman functions, are to be smeared with suitable test functions to generate observables. We would therefore normally define a micro-causality condition by smearing the Wightman function with test functions of spacelike separated support. However, to do so, the Wightman functions need to be tempered distributions, so that we can use the usual Schwartz space of test functions. Physically this amounts to the correlation functions in momentum space growing at most polynomially.

In the non-local theory, where momentum space correlators grow exponentially, the Wightman functions are rather singular distributions and the allowed space of test functions is restricted. In this case test functions are required to be real analytic in position space, which precludes local observables (cf. [27] for references and rigorous arguments). This prevents us from defining a micro-causality condition in the usual way, as we cannot smear with test functions of spacelike separated support.

In [27], it was proposed that we can still study micro-causality in these cases, by considering the analytic structure of the Wightman functions at points where they make sense as functions. Usually micro-causality implies permutation symmetry of Wightman functions: for example, $W(x_1, x_2) = W(x_2, x_1)$ for spacelike separated points x_1 and x_2. However, for the more singular Wightman functions in a non-local theory, the region of analytic behaviour is restricted. In the case of little string theories, [27] observed that the Wightman function is analytic only for $(x_1 - x_2)^2 > l_s^2$, and proposed that the micro-causality condition be replaced by requiring permutation symmetry of Wightman functions in this restricted region. The usual commutativity condition is modified by the presence of poles in the region $0 < (x_1 - x_2)^2 < l_s^2$.

While a detailed analysis of the analyticity properties of the non-commutative gauge theory Wightman functions is beyond the scope of the current paper, we believe that it should be possible to construct an analogous argument to determine the micro-causality properties associated with the gauge-invariant operators $\mathcal{O}(k)$, relating the restriction to the light-wedge (3.12) seen from the spacetime point of view to the fact that the momentum space correlator (3.13) is growing only along the non-commutative directions. Recent discussions of microcausality conditions in non-commutative theories can be found in [28, 29].

It is also useful to note that agreement is expected: the micro-causality conditions for an interacting theory on the boundary ought to be captured by a free-field theory in the bulk in the large N limit (where the supergravity approximation is expected to valid), because the correct micro-causality conditions in the field theory stem from the quantum 1PI propagator. This might seem a bit puzzling in the context of the previous discussion of non-commutative scalar field theories, as we see (in appendix A) that the effect of non-commutativity arose from non-planar diagrams in the field theory. However, these non-planar contributions are associated with the non-commutative interactions, and are not suppressed by large-N power counting; in their absence, the spacetime dual for non-commutative $\mathcal{N} = 4$ SYM would have been just $\mathrm{AdS}_5 \times \mathbf{S}^5$, rather than the spacetime (2.1) with its complicated asymptotics.

4. Generating spacetimes by null Melvin twist

We next want to extend the discussion to the spacetimes dual to lightlike non-commutative field theories. We will begin by presenting the derivation of these solutions, using a particular solution generating technique in supergravity, called the null Melvin twist [16, 17]. This is slightly more complicated than the twist used to obtain the geometry in the spacelike case, and will add a B-field oriented along a lightlike direction. While some of the solutions we are interested in have been obtained previously in the literature [18], re-deriving them in this way will simplify the discussion of generalisations. A nice summary of the solution generating scheme we use and some classifications of solutions can be found in [30].

4.1 The null Melvin twist

The solution generating scheme is:

1. Start with a solution of IIA/IIB supergravity, with a non-compact translationally invariant direction; the corresponding Killing vector will be taken to be $\left(\frac{\partial}{\partial y}\right)^a$.

2. Boost the geometry in the y direction by an amount γ. Note that this is just a coordinate change, which effectively adds momentum charge to the solution we start with.

3. Now perform a T-duality along y, to get to a solution of IIB/IIA supergravity.

   The above steps will typically add a fundamental string charge to the solution we start with. However, if the Killing field $\left(\frac{\partial}{\partial y}\right)^a$ can be paired with a timelike Killing field $\left(\frac{\partial}{\partial t}\right)^a$ (so that our starting solution has SO(1, 1) isometry), then no charge is added. Instead Step **2** is trivial as the geometry is boost invariant. Step **3** is then just a diagonal T-duality.

4. *The Twist:* Assume that we have in addition to the Killing field $\left(\frac{\partial}{\partial y}\right)^a$, some other rotational or translational isometries. We would like to perform a non-diagonal T-duality by combining these isometries with that generated by $\left(\frac{\partial}{\partial y}\right)^a$. Schematically, denoting the one-forms dual to the additional isometries by σ, we perform a twist (a coordinate transformation) by replacing

$$\sigma \to \sigma + 2\,\alpha\, dy\,. \tag{4.1}$$

   Here α parameterises the amount of twisting.

5. We now T-dualize the geometry back to IIA/IIB along y. The twist followed by the T-duality is effectively a non-diagonal T-duality.

6. Boost the solution by $-\gamma$ along y. The purpose of this boost is in part to undo the original boost performed.

7. Now, we perform a double scaling limit, wherein the boost γ is scaled to infinity and the twist α to zero keeping

$$\beta = \frac{1}{2}\,\alpha\, e^{\gamma} = \text{fixed}\,. \tag{4.2}$$

The null Melvin twist transformations in steps 4 through 7 can be thought of as converting the string solution into a fluxbrane, followed by a boost and scaling to end up with a null isometry.

4.2 Null Melvin twist of the D3-brane solution

Let us apply this transformation to the D3-brane geometry. The extremal D3-brane geometry is a solution of type-IIB supergravity with metric:

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + \sqrt{H(r)}\,\left(dr^2 + r^2\, d\Omega_5^2\right), \tag{4.3}$$

with

$$H(r) = 1 + \frac{R^4}{r^4}\,. \tag{4.4}$$

The metric (4.3) is supported by a five-form flux. Since the flux turns out to be insensitive to the null Melvin twist we will not write it explicitly.

The geometry (4.3) has a SO(1, 1) Lorentz symmetry along the world-volume directions (t, y), and so we will not have any charge added during the first two steps. The twisted T-duality implemented in steps 4 and 5 requires us to pick a direction to do the twisting. The simplest choice turns out to be the translationally invariant directions $\left(\frac{\partial}{\partial x^1}\right)^a$ and $\left(\frac{\partial}{\partial x^2}\right)^a$. Performing the twist as

$$dx_1 \;\to\; dx_1 + \alpha\, dy\,, \qquad dx_2 \;\to\; dx_2 + \alpha\, dy\,, \tag{4.5}$$

we obtain the following geometry:

$$
\begin{aligned}
ds^2 &= -\beta^2 H^{-\frac{3}{2}} (dt+dy)^2 + \frac{1}{\sqrt{H}} \left(-dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + \sqrt{H} \left(dr^2 + r^2\, d\Omega_5^2\right) , \\
B &= \beta\, \frac{1}{H} (dt+dy) \wedge (dx_1 + dx_2) .
\end{aligned} \tag{4.6}
$$

Apart from the NS-NS 3-form flux written above the metric is in addition supported by a five-form flux (which is the same as for the original D3-brane solution (4.3)). The sequence of operations **1** to **7** maps a constant dilaton solution back to a constant dilaton solution for this particular case. Note that the solution (4.6) is asymptotically flat.

In deriving (4.6) we have used the translational isometries $\left(\frac{\partial}{\partial x^i}\right)^a$ to perform the twisted T-duality. We could just as well have used some angular isometries, such as the rotation isometry in the (x^1, x^2) plane, or isometries of the $\mathbf{S}^5$. In such cases, we would generate asymptotically plane wave, rather than asymptotically flat, spacetime; however, since this asymptotic region will be lost once we take the decoupling limit, the interesting causal features of the plane wave (such as 1-dimensional boundary or non-global hyperbolicity) would not enter our story. These cases are briefly discussed in appendix C, although the essential physical point of interest is exemplified adequately by (4.6).

4.3 Near-horizon geometry of null Melvin twisted D3-brane

Our main interest is to identify interesting generalisations of the AdS/CFT correspondence, so we now consider an appropriate decoupling limit of the geometry (4.6), to obtain a new duality relating the low energy degrees of freedom on the twisted D3-brane to a supergravity geometry. For the D3-brane geometry (4.3), we know that the near horizon limit corresponding to decoupling the closed string modes from the open string modes on the D3-brane is obtained by dropping the 1 in the harmonic function $H(r)$ given in (4.4) [31]. For the solution (4.6), we want to ensure that the effect of the twist survives in the decoupling limit. Thus the appropriate limit is analogous to the Seiberg-Witten scaling for non-commutative field theories [32]. This again amounts to dropping the 1 from the harmonic function $H(r)$ (4.4), now in the new metric (4.6). The resulting geometry is given by

$$
\begin{aligned}
ds^2 &= -\beta^2\, \frac{r^6}{R^6}\, du^2 + \frac{r^2}{R^2} \left(-2\, du\, dv + dx_1^2 + dx_2^2\right) + \frac{R^2}{r^2} \left(dr^2 + r^2\, d\Omega_5^2\right) , \\
B &= \beta\, \frac{r^4}{R^4}\, du \wedge (dx_1 + dx_2) ,
\end{aligned} \tag{4.7}
$$

where we have introduced light cone coordinates $u = t + y$ and $v = t - y$. In addition to the NS-NS B-flux there is also a five-form flux, which is identical to that supporting the $\mathrm{AdS}_5 \times \mathbf{S}^5$ solution. The metric only differs from the $\mathrm{AdS}_5 \times \mathbf{S}^5$ metric written in Poincaré coordinates by the term proportional to β^2. This solution was previously derived in [18]. They considered the solution to type-IIB supergravity that was known to be dual to $\mathcal{N} = 4$ SYM deformed by spacelike non-commutativity [14, 15], and boosted it to derive (4.7). As in the case of spacelike non-commutativity, it is difficult to define a conformal boundary for this spacetime. The correspondence to the field theory is derived by thinking of the field theory as living on the D3-branes that source the full geometry (4.6), and taking the low-energy limit.

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We now show that this solution is non-singular, and non-distinguishing. We show that it is non-singular by observing that all the curvature invariants for (4.7) are the same as for $AdS_5 \times \mathbf{S}^5$, and that the spacetime is geodesically complete. The additional contributions to the curvature from the term proportional to β^2 will involve $\left(\frac{\partial}{\partial u}\right)^a$, so they will not change the curvature invariants. Geodesic completeness can be shown either by observing that the metric is conformal to a pp-wave,

$$\begin{aligned} ds^2 &= r^2 \left(-2\, du\, dv - \beta^2\, r^4\, du^2 + dx_1^2 + dx_2^2 + \frac{dr^2}{r^4} + \frac{1}{r^2}\, d\Omega_5^2 \right), \\ &= \frac{1}{z^2} \left(-2\, du\, dv - \frac{\beta^2}{z^4}\, du^2 + dx_1^2 + dx_2^2 + dz^2 + z^2\, d\Omega_5^2 \right) \end{aligned} \tag{4.8}$$

(where we have set $R = 1$ and in the second line used $r = 1/z$), and appealing to the results of [33], or by explicitly studying the geodesics. We will discuss timelike geodesics in the geometry (4.8) in section 6. The essential point is that the motion in the radial direction is confined to a finite range and inertial observers are unable to escape towards the asymptotic region (cf., (6.3), where it is clear that geodesics cannot access the region $z = 0$, or equivalently $r = \infty$, due to a potential barrier). Thus, though we have diverging curvatures in the large r region, the spacetime is non-singular.

To show that the spacetime is non-distinguishing, it is easiest to use the representation (4.8) of the metric as conformal to a pp-wave, and use the results of [11, 12]. Since causal properties are invariant under conformal transformations, the causal character of our spacetime is determined by that of the pp-wave spacetime within the parenthesis in (4.8). In [11, 12] it was shown that these pp-wave spacetimes are non-distinguishing. Causal curves that connect a point $P = (u_0, v_0, x_0^i, z_0)$ in the spacetime to any point $Q = (u_0 + \varepsilon, v_1, x_1^i, z_1)$, with $\varepsilon > 0$ and arbitrary values of (v_1, x_1^i, z_1) were explicitly constructed. This shows that the causal future of P is the entire region $u \geq u_0$, as depicted in figure 1. In the terminology of [11], the metric (4.8) written in the r coordinate is conformal to a super-quadratic pp-wave, as g_{uu} grows faster than a quadratic for large r. We give an explicit construction of the causal curve from P to Q (in the r coordinates) in appendix B.

5. The holographic dual field theory

As can be seen either from the lightlike nature of the B-field in the spacetime geometry (4.6) or from the alternative derivation[2] of this geometry in [18], the dual field theory description of the geometry (4.8) is a $d = 4$, $\mathcal{N} = 4$ non-commutative Super Yang-Mills (NCYM) on $\mathbf{R}^{3,1}$ with a constant lightlike non-commutativity parameter $\theta^{v\,i} = \beta$ for $i = 1, 2$.

5.1 Well behavedness of observables in the field theory

Such lightlike non-commutativity was first discussed in [19], where it was shown that theories with lightlike non-commutativity behave like field theories with non-local interactions.

[2]In [18], the solution was obtained by starting from the holographic dual to NCYM with spacelike non-commutativity and then boosting.

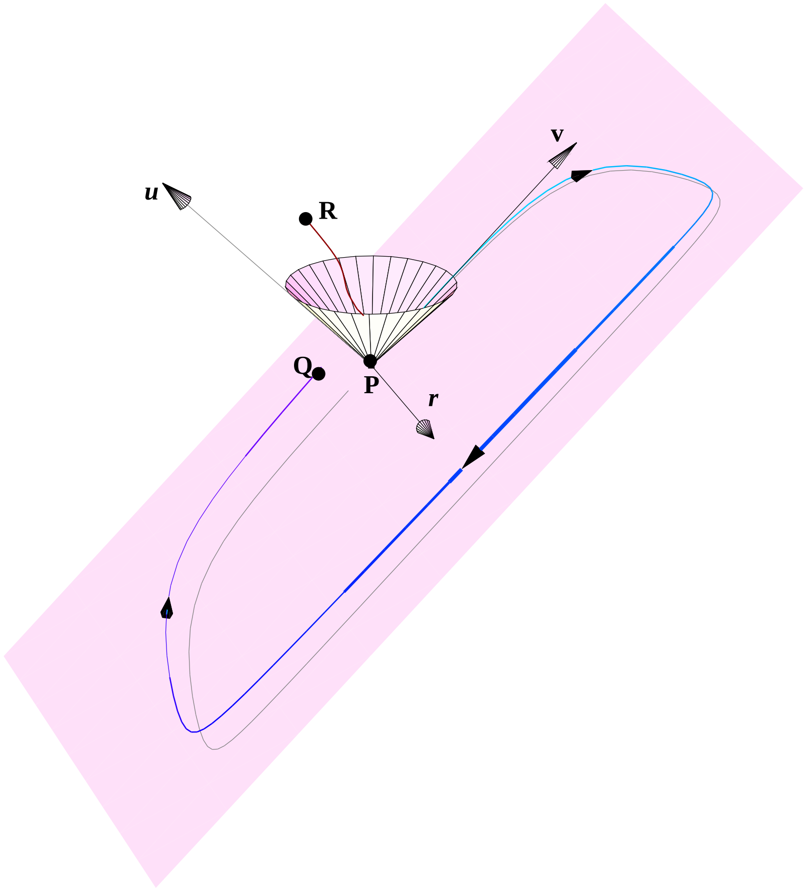

Figure 1: Light cone structure in the non-distinguishing spacetime (4.8). From point **P** we can reach point **R** by a usual causal curve. To reach point **Q** which is spacelike separated in the Minkowski light cone, we need to use the asymptotic structure of the spacetime. By using curves that venture out further in r we can arrange for the causal curve to return arbitrarily close to the starting point. As a result the future of the point **P** is the part of the spacetime above the constant u plane shown above.

This is in contradistinction with the case of timelike non-commutativity, where we have a string theoretic behaviour due to the non-decoupling of excited open string oscillators. The crucial difference between the two cases is the following [19]: in the case of timelike non-commutativity, we start from a D-brane background and turn on electric fields on the brane world-volume [34]. There is however a critical limit to the electric field (T-dual to there being a maximum velocity – speed of light), and at the critical point the open strings become effectively tensionless. The timelike theory is defined by zooming in on this critical region, and as a result one ends up with a finite effective string tension. In the case of light-like non-commutativity, we turn on both electric and magnetic fields, and it is easy to see that this combination doesn't lead to a critical electric field. Alternatively, one can think of the lightlike theory as a boosted version of the spacelike theory, where there is clearly no critical behaviour. Thus, this dual is a well behaved field theory with non-local interactions. Hence it is quite surprising that the dual geometry (4.8) exhibits some causal pathology.

To understand better the physics of lightlike non-commutative field theories we can consider observables in these theories, such as the correlation function of local operators $\mathcal{O}(p)$. These can be calculated at strong 't Hooft coupling using the using the usual bulk-boundary correspondence. It transpires that massless minimally coupled scalar fields in the bulk geometry (4.8) satisfy a Mathieu equation which is very similar to that obtained in [15] for the geometry (2.1). Of course the parameters appearing in the equation have a different dependence on momenta, as should be expected given the different symmetries of the geometries. This implies that the correlation functions are schematically similar to the spacelike case. Likewise, one can consider gauge invariant observables built with open Wilson lines as discussed in [24] and check that these are also well behaved and in fact will be exponential in the momenta. In summary, there is no reason from a field theory perspective that lightlike non-commutative field theories should be pathological, which makes the bizarre causal structure of the dual spacetime geometry all the more interesting.

5.2 Genericity of non-distinguishingness for lightlike non-commutative field theories

We have shown above that the supergravity background dual to lightlike non-commutative $\mathcal{N}=4$ SYM is a non-distinguishing spacetime. We will now show that any well defined field theory will produce a non-distinguishing holographic dual when we deform it by adding some lightlike non-commutativity.

Consider any field theory defined in the usual Wilsonian sense, by its ultraviolet modes. From the UV/IR correspondence for gauge/gravity duality [35], we know that the field theory ultra-violet corresponds to the infra-red region of the supergravity dual. For field theories on $\mathbf{R}^{3,1}$ one can without loss of generality assume that the holographic dual geometry is of the following warped product form:

$$ds^2 = A(r)^2 \left(-dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + dy_6^2 \,, \tag{5.1}$$

where r is an effective radial coordinate on the transverse space with metric dy_6^2. If the field theory approaches a conformal fixed point in the UV, the large r behaviour will be asymptotically $\mathrm{AdS}_5 \times X^5$ for some compact Einstein manifold X^5. Our argument will however be general enough to include more exotic field theories which do not arise from nice conformal fixed points in the UV, such as the Klebanov-Strassler cascade [36].

We would like to turn on lightlike non-commutativity in the field theory. We have seen above that this is achieved by a Null Melvin Twist on the dual geometry (5.1). The Poincaré invariance of the field theories on $\mathbf{R}^{3,1}$ implies that there are no fluxes that prevent us from carrying through the steps involved in the duality chain. A null Melvin twist of (5.1) will lead to the following geometry:

$$ds^2 = A(r)^2 \,\left(-2\,du\,dv - \beta^2\,A(r)^4\,du^2 + dx_1^2 + dx_2^2\right) + dy_6^2 \,. \tag{5.2}$$

The question is then what the effect of the additional du^2 term is on the causal structure; this depends on the particular form of $A(r)$. For field theories with a conformally invariant fixed point in the UV, the large r behaviour of $A(r)$ is $A(r) \sim e^{ar}$, giving the same kind of asymptotics as in the particular case we studied, and hence implying that for such cases, the metric (5.2) is non-distinguishing. The same is true for the $A(r)$ appropriate to the Klebanov-Strassler geometry [36], so non-commutative deformations of this theory also have non-distinguishing duals, see appendix C for the explicit metric.

Thus, we see that the UV behaviour of the field theory is responsible for the dual supergravity background being causally ill-behaved. It follows that infrared modifications of the field theory will not remove the causal pathologies, as they do not change the asymptotics of the dual spacetimes. For example, the thermal version of the non-commutative theory will be dual to a spacetime with a black hole in it, but this will still be non-distinguishing. For completeness, we present the metric for the thermal version of the lightlike non-commutative $\mathcal{N} = 4$ SYM in appendix C.

6. Causality in lightlike non-commutative theories

We now attempt to understand the origin of the non-distinguishing character of the spacetime (4.8) from the field theory perspective. As with the case of spacelike non-commutativity, it will be instructive to first understand the behaviour of the bulk light cone in the asymptotic region. We will then proceed to look at the micro-causality condition in field theory and show that the perturbative micro-causal light cone is modified so as to be consistent with the characteristics of the bulk spacetime. As in section 3, we consider a simple scalar field theory as a model, but it seems reasonable to expect that this micro-causal structure will be independent of the details of the particular field theory we consider.

6.1 Lightlike non-commutativity: bulk light cone

Our derivation of the bulk light cone for the lightlike non-commutative field theories will proceed in a fashion analogous to the spacelike case discussed in section 2. We will focus once again on the properties of the bulk light cone in the asymptotic region of the spacetime, which for (4.8) will be the region $r \to \infty$ or $z \to 0$.

Let us consider the induced metric on a surface of fixed $r = r_0$, so as to discern the causal properties in the cut-off lightlike non-commutative field theory. This induced metric is (in what follows we will ignore the $\mathbf{S}^5$ directions)

$$ds^2 \approx r_0^2 \left(-2\, du\, dv - \beta^2\, r_0^4\, du^2 + dx_1^2 + dx_2^2\right), \tag{6.1}$$

which of course is the metric on flat $\mathbf{R}^{3,1}$ once we rescale the coordinates appropriately. However, if we restrict to homogeneous rescaling of the coordinates, the metric (6.1) degenerates to a one dimensional metric due to the dominance of g_{uu} for large r_0. We then expect the bulk light cone to degenerate to a galilean causal structure, in which any two points are causally related unless $\Delta u = 0$.

The above argument for the asymptotic bulk light cone can be made precise by considering the two-point function. For the metric (4.8), the free scalar wave equation is still the Mathieu equation, making explicit determination of the propagator tricky. We will therefore concentrate again on a geodesic approximation to the propagator. Let us begin by considering timelike geodesics in the geometry (4.8),

$$-1 = \frac{1}{z^2}\left(-2\,\dot{u}\,\dot{v} - \frac{\beta^2}{z^4}\,\dot{u}^2 + \dot{x}_1^2 + \dot{x}_2^2 + \dot{z}^2\right), \tag{6.2}$$

where $\cdot \equiv \frac{d}{d\tau}$, with τ the affine parameter along the geodesics. The Killing symmetries $\left(\frac{\partial}{\partial u}\right)^a$, $\left(\frac{\partial}{\partial v}\right)^a$ and $\left(\frac{\partial}{\partial x^i}\right)^a$ determine conserved energies and momenta (p_u, p_v, p_i), respectively. The geodesic motion reduces to a classical problem for the zero energy trajectories of a particle in an effective potential[3] $V_{\rm eff}(z)$:

$$\begin{aligned}
\dot{z}^2 + V_{\rm eff}(z) &= 0\,, \\
\dot{u} = -p_v\,z^2\,, \qquad \dot{v} &= -p_u\,z^2 + \frac{\beta^2\,p_v}{z^2}\,, \qquad \dot{x}_i = p_i\,z^2\,, \\
V_{\rm eff}(z) &= z^2 + \left(p_1^2 + p_2^2 - 2\,p_u\,p_v\right) z^4 + p_v^2\,\beta^2\,.
\end{aligned} \tag{6.3}$$

Let us consider geodesics that travel an infinitesimal distance in the radial (z) direction. Choose $z = z_0\,(1+\varepsilon)$ with $\varepsilon \ll 1$ such that $V_{\rm eff}(z_0) = 0$. Then we have

$$z_0^2\,\dot{\varepsilon}^2 \approx 2\,z_0^2\,\varepsilon\left(1 + 2\,\frac{\beta^2\,p_v^2}{z_0^2}\right) \equiv 2\,\varepsilon\,\alpha\,. \tag{6.4}$$

This allows us to approximately integrate the geodesic equations to obtain,

$$\Delta u = p_v\,z_0^3\,\frac{\sqrt{2\,\varepsilon}}{\sqrt{\alpha}}\,, \qquad \Delta v = \left(p_u - \frac{\beta^2\,p_v}{z_0^4}\right) z_0^3\,\frac{\sqrt{2\,\varepsilon}}{\sqrt{\alpha}}\,, \qquad \Delta x_i = p_i\,z_0^3\,\frac{\sqrt{2\,\varepsilon}}{\sqrt{\alpha}}\,, \tag{6.5}$$

It is obvious from (6.5) that one can make the distance travelled by the geodesics in individual directions large by choosing appropriate momenta. However, the motion is confined to the region in the immediate vicinity of the classical bulk light cone, as

$$-2\,\Delta u\,\Delta v - \frac{\beta^2}{z_0^4}\,\Delta u^2 + \Delta x_1^2 + \Delta x_2^2 \approx -\frac{2\,\varepsilon}{\alpha}\,z_0^4 \ll 1\,. \tag{6.6}$$

Thus, bulk geodesics can relate points inside the effective light cone given by

$$-2\,\Delta u\,\Delta v - \frac{\beta^2}{z_0^4}\,\Delta u^2 + \Delta x_1^2 + \Delta x_2^2 = 0\,, \tag{6.7}$$

which limits as $z_0 \to 0$ to the galilean invariant light wedge. Causal properties are then determined by just the value of u coordinate. We see that the point $P = (u_0, v_0, x_0^i)$ is in the past of $Q = (u_1, v_1, x_1^i)$ if $u_0 < u_1$ irrespective of the other coordinates.

[3]From the form of $V_{\rm eff}(z)$ it is clear that geodesics never reach $z = 0$ for $p_v \neq 0$, thereby preventing inertial observers from being subject to large tidal forces resulting from the diverging curvatures in that region. This is the hitherto alluded to characteristic that demonstrates geodesic completeness of (4.8).

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6.2 Micro-causality in lightlike non-commutative field theories

Let us now study the micro-causal light cone for this case of lightlike non-commutativity, following the discussion of the spacelike case given earlier. First of all, we can use the simpler argument of [6] to suggest the appropriate micro-causality condition for this case. Suppose we have a field theory in $\mathbf{R}^{d,1}$ with $\theta^{ui} \neq 0$, where we use lightcone coordinates (u, v, x^i) for $i = 1, 2, \ldots, d-1$. This non-commutative deformation will break the Lorentz group $\mathrm{SO}(d,1)$ down to a galilean group. The natural galilean invariant micro-causal relation to consider is

$$\langle\, [\Phi(x), \Phi(y)]_{\pm}\, \rangle = 0 \qquad \text{for } u_x = u_y\,. \tag{6.8}$$

We can use the argument of [7] as briefly reviewed in section 3 to show that the lightlike non-commutative field theory in fact has such a micro-causality condition, while a local quantum field theory with galilean invariance will still have the usual commutation relations (3.1).

For sake of simplicity we consider the non-commutative φ^3 theory introduced in (3.4); we will calculate the perturbative behaviour of the field commutator expectation value to probe the micro-causality condition. In fact, the calculation proceeds in a manner similar to the spacelike non-commutative case, but for a few essential differences in the details of the kernels. As in that case there is no difference from commutative φ^3 theory at the tree level i.e., for $\mathcal{M}^{(0)}$. Once again interesting effects show up at the one-loop contribution to the commutator, thanks to the contribution from the non-planar diagrams. The calculation is outlined in appendix A. We find that the non-planar graphs give us one-loop contribution

$$G_2^{(np)}(\ell_1, \ell_2, x) \sim \int_0^\infty d\ell_1\, d\ell_2\, \exp\left(i \frac{\vec{\theta}\cdot\vec{\theta}}{2\,u_x}\, \ell_1\, \ell_2\, (\ell_1 + \ell_2) - i\, \frac{m^2\, u_x}{2} \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \right). \tag{6.9}$$

In writing the above we have retained only the dominant contribution at large momentum (for this is the part that determines the convergence properties) and $\vec{\theta}$ is a vector with components θ^{ui}. We have furthermore

$$\mathcal{M}_{np}^{(2)}(x) \sim \int_0^\infty d\ell_1 \int_0^\infty d\ell_2\, G_2^{(np)}(\ell_1, \ell_2, x) - \int_0^\infty d\ell_1 \int_0^\infty d\ell_2\, G_2^{(np)}(\ell_1, \ell_2, -x)\,, \tag{6.10}$$

and as usual, $\mathcal{M}_{planar}^{(2)}(x)$ is the same as in the commutative field theory.

In order to ascertain the micro-causality condition, we need to figure out the domain where $\mathcal{M}_{np}^{(2)}$ is guaranteed to vanish. This can be done by looking at the analytic properties of $G_2^{(np)}(\ell_1, \ell_2, x)$. Assuming without loss of generality $u_x > 0$, we see that $G_2^{(np)}(\ell_1, \ell_2, x)$ will be convergent only for $\ell_{1,2} \to -i\,\infty$. However, for $\ell_{1,2} \to -i\,0$ we encounter a divergence from the term proportional to m^2 in the exponential. Hence we are forced to conclude that there is no domain in the complex $\ell_{1,2}$ plane where the integral of $G_2^{(np)}(\ell_1, \ell_2, x)$ is absolutely convergent. This in particular implies that we are generically going to encounter a non-vanishing value of $\mathcal{M}^{(2)}$. The only special case is when $u_x = 0$ when we expect $\mathcal{M}^{(2)}$ will indeed vanish. This is not apparent from (6.9) and (6.10), since these are written in a light cone quantization scheme. Intuitively, one expects this to arise simply from the fact that this is indeed the 'equal time' commutation relation for light cone quantization. To summarise, the micro-causality condition for this lightlike non-commutative field theory is indeed as given by (6.8).

As we have emphasized in the spacelike non-commutative case, it would be challenging to extend this computation to the non-commutative Yang-Mills theory which is actually dual to the bulk spacetime. Once again, the appropriate framework for exploring these ideas would be to look at the details of the analytic behaviour of the Wightman functions. However, as argued in the spacelike case, the micro-causal light cone for the field theory ought to be in agreement with the causal properties of the supergravity dual. So, it is again encouraging to find that the results in the scalar field theory exhibit qualitative agreement with the bulk spacetime. Since the bulk behaviour was intimately associated with the non-distinguishing character of this spacetime, we might even say that this non-distinguishing character is not only consistent in string theory, but in fact necessary to reproduce this behaviour in the dual field theory.

7. Discussion

We have studied modifications of the causal structure of the bulk spacetime in the AdS/CFT correspondence, by considering the changes in the geometry dual to a non-commutative deformation of the field theory. We pointed out that these deformations cause radical changes in the bulk causal structure. For spacelike non-commutativity, the light cone in the boundary directions asymptotically becomes independent of the non-commutative directions. For lightlike non-commutativity, the modification is even stronger, producing a non-distinguishing spacetime, in which points are always connected by a causal curve unless they have the same value of 'light-cone time'.

We compared the asymptotic bulk light cone to the micro-causal light cone in a scalar non-commutative field theory, following [6, 7]. In this toy example we were able to show that the micro-causal light cone was in complete agreement with the bulk predictions. As we have already remarked, we think it reasonable to expect this agreement to extend to $\mathcal{N} = 4$ non-commutative gauge theory, and have suggested a possible way to infer this rigorously.

The essential issue in considering non-commutative gauge dynamics is the absence of local gauge invariant observables in position space. However, any bulk calculation of correlation function in strongly coupled non-commutative gauge theory will exhibit the deformed light cone structure. From the point of AdS/CFT correspondence it is imperative that this structure be reproduced from the field theory side as well. In fact, as discussed in [24, 37] there is an excellent agreement between the result for the correlation function of gauge invariant open Wilson loop operators in the field theory (obtained by summing up ladder diagrams) and the bulk prediction. Furthermore, given the nature of observables in the non-commutative gauge theory, this agreement is expected to be universal. Since the bulk two point function is sensitive to the deformation of the light cone, it seems quite natural to expect the same in the field theory.

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These results provide a new connection between the causal structure in the bulk and in the boundary. However, the full encoding of the bulk causal structure in the dual field theory description remains an important open problem. In particular, we have not addressed the very interesting question of how the bulk causal structure is encoded when the Poincaré invariance in broken only in the interior (as in black hole solutions).

The discovery that a non-distinguishing spacetime is related via the AdS/CFT duality to a well-defined quantum field theory is of interest in its own right. It suggests that such geometries may be more respectable as string theory backgrounds than one might have expected. The usual objection to non-distinguishing backgrounds, that a small deformation can convert them into a solution with closed timelike curves, is here circumvented by the fact that the non-distinguishing character is a consequence of the behaviour of the light-cone in the asymptotic region, where corrections are suppressed. That is, although there are almost closed timelike curves, they do not remain within a compact region of the spacetime. These solutions deserve further investigation.

Acknowledgments

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A. Derivation of light-wedge

We present in this appendix a short derivation of the micro-causality condition in non-commutative field theories. Further details can be found in [7].

As discussed in section 3, we are interested in calculating the matrix element of the field commutator in perturbation theory. The field theory we consider will be non-commutative φ^3 theory in $\mathbf{R}^{5,1}$ with lagrangian (3.4). In what follows bold face letters will denote spatial vectors $\{\mathbf{p}, \mathbf{x}\}$ etc., in $\mathbf{R}^5 \subset \mathbf{R}^{5,1}$. We will also have use for light cone coordinates and the vectors transverse to the light cone directions will be denoted as $\{\vec{p}, \vec{x}\}$.

Working within the framework of canonical quantization, one can write the interaction picture field $\varphi(x)$ as

$$\varphi(x) = \int \frac{d^5\mathbf{p}}{(2\,\pi)^5} \, \frac{1}{\sqrt{2\,\omega_p}} \left(a_p \, e^{-i\,p.x} + a_p^\dagger \, e^{i\,p.x} \right), \tag{A.1}$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$.

As usual it is possible to pass between the interaction and Heisenberg pictures:

$$\varphi_H(x) = U^\dagger(t, t_0) \, \varphi(x) \, U(t, t_0) \, , \qquad U(t_1, t_2) = \mathrm{T} \exp \left(i \, \frac{g}{3!} \int_{t_1}^{t_2} dt \int d^5x \, \varphi \star \varphi \star \varphi \right). \tag{A.2}$$

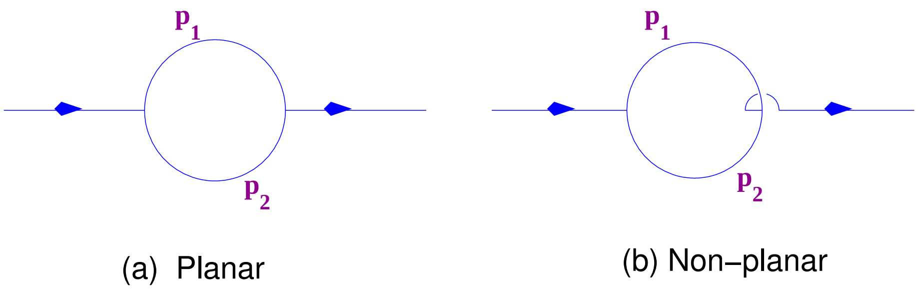

Figure 2: Feynman diagrams contributing to $\mathcal{M}^{(2)}$. Due to the non-commutative interaction we have both planar and non-planar graphs.

We will make use of this to evaluate the matrix element (3.5) and consider the states $|\,\alpha\,\rangle$ and $|\,\beta\,\rangle$ obtained by evolving the perturbative vacuum $|\,0\,\rangle$ by the evolutions operator $U^{\dagger}(t_1, t_0)$.

For the $\varphi^3_\star$ theory the quantities of interest are the tree-level term $\mathcal{M}^{(0)}$ and the one-loop term $\mathcal{M}^{(2)}$. The former is just the usual free field propagator; the presence of non-commutative interactions play no interesting role for the quadratic terms in the lagrangian (3.4). In fact, we have

$$\mathcal{M}^{(0)}(x = x_1 - x_2) = \int \frac{d^5\mathbf{p}}{(2\,\pi^5)}\,\frac{1}{2\,\omega_p}\,e^{-i\,p.x} - \int \frac{d^5\mathbf{p}}{(2\,\pi^5)}\,\frac{1}{2\,\omega_p}\,e^{+i\,p.x}\,, \tag{A.3}$$

which may be written in the form (3.7), using the identity

$$\int \frac{d^5\mathbf{p}}{\omega_p} = \int_0^\infty \frac{dp^+}{p^+}\,d^4\vec{p}\,\,|_{p^- = \frac{\vec{p}^2 + m^2}{2\,p^+}}\,. \tag{A.4}$$

In writing the above we have passed over to light cone representation of momenta for convenience. Hence we obtain

$$\int \frac{d^5\mathbf{p}}{(2\,\pi^5)}\,\frac{1}{2\,\omega_p}\,e^{-i\,p.x} = 2\,(2\,\pi)^5 \int_0^\infty \frac{dp^+}{p^+}\,d^4\vec{p}\,\exp\left(i\,p^+\,v_x + \frac{\vec{p}^2 + m^2}{2\,p^+}\,u_x - i\,\vec{p}.\vec{x}\right). \tag{A.5}$$

Completing the gaussian integrals over $\vec{p}$ in (A.5) we find the integral representation of $\mathcal{M}^{(0)}$ with the kernel $G_0(p^+ = \ell, x)$ as given in (3.9).

From this representation of $\mathcal{M}^{(0)}(x)$ one can ascertain the domains where the commutator is required to vanish. As discussed in section 3 the strategy is to ascertain the possible contour rotations for the $p^+ = \ell$ integral. Depending on the convergence of the integrals of $G_0(\ell, \pm x)$, the two terms contributing to $\mathcal{M}^{(0)}(x)$ mutually cancel. This is where we see the origin of the micro-causality condition in the field theory. The contour rotations that lead to cancellation of the two integrals work only for spacelike separated points and not for points $x^\mu_{1,2}$ which are in causal contact.

The calculation of $\mathcal{M}^{(2)}$ proceeds in a similar fashion. We get contributions for the non-commutative φ^3 theory from two kinds of one-loop diagrams: planar and non-planar. The planar diagrams are very similar to those of the commutative theory and do not change the micro-causal structure. The non-planar diagrams introduce extra momentum dependent phase factors through the $\varphi_\star^3$ term. These phase factors play an important role in modifying the properties of $G_2(\ell_i, x)$. The relevant Feynman diagrams are shown in figure 2.

We will concentrate exclusively on the non-planar contribution to the commutator. The phase factor originating from the $\varphi_\star^3$ interaction for the non-planar graph is

$$P(p_1, p_2) = e^{i\, p_1^\mu\, \theta_{\mu\nu}\, p_2^\nu}\,, \tag{A.6}$$

Ignoring numerical coefficients we can write

$$\mathcal{M}_{np}^{(2)} \sim \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 \left[G_2^{(np)}(\ell_1, \ell_2, x) - G_2^{(np)}(\ell_1, \ell_2, -x)\right], \tag{A.7}$$

where

$$G_2^{(np)}(\ell_1, \ell_2, x) = \exp\left(i\, \frac{x^2}{2u_x}\,(\ell_1 + \ell_2) - i\,\frac{m^2 u_x}{2}\left(\frac{1}{\ell_1} + \frac{1}{\ell_2}\right)\right) H_2(\ell_1, \ell_2, x)\,, \tag{A.8}$$

is of the general form described in (3.8), and the kernel $H_2(\ell_1, \ell_2, x)$ is

$$H_2(\ell_1, \ell_2, x) = \int \frac{d^4\vec{p}_1\, d^4\vec{p}_2}{(2\,\pi)^{10}}\, e^{-i\,\frac{u_x}{2\,\ell_1}\left(\vec{p}_1 - \ell_1\,\frac{\vec{x}}{u_x}\right)^2 - i\,\frac{u_x}{2\,\ell_2}\left(\vec{p}_2 - \ell_2\,\frac{\vec{x}}{u_x}\right)^2}\, P(p_1, p_2) Q\,. \tag{A.9}$$

Q is a function of the momenta that grows like a polynomial and hence doesn't play any role in our discussions. Readers interested in its precise form can consult [7]. In deriving the above we have made use of the identity (A.4) given above to convert the momentum integrals over $d^5\mathbf{p}_{1,2}$ to integrals over $d^4\vec{p}_{1,2}$ and over the light cone component labeled here as $\ell_{1,2}$ respectively. Note that while we have lumped a lot of the details of the interaction in the factor Q, we are keeping the phase factor $P(p_1, p_2)$ explicitly, since it is an exponential function of the momenta.

We now analyze the large momentum behaviour of (A.9) by completing the squares and performing the resulting integral (setting $Q = 1$). So far our discussion has been independent of the precise nature of the non-commutative interaction. We will see that there is a distinction between the spacelike non-commutative and lightlike non-commutative cases. The essential point is that the large momentum behaviour of (A.9) depends on the nature of the non-commutative interaction, and so we will treat these two cases separately.

A.1 Spacelike non-commmutativity

For spacelike non-commutative theories we will choose $\theta^{\mu\nu} \neq 0$ only along the spatial directions transverse to our light cone coordinates, thus having $\theta^{ij} \neq 0$. So we can write (A.9) as

$$\begin{aligned} H_2(\ell_1, \ell_2, x) &\sim \int d^4\vec{p}_1\, d^4\vec{p}_2\, e^{-i\,\frac{u_x}{2\,\ell_1}\left(\vec{p}_1 - \ell_1\,\frac{\vec{x}}{u_x}\right)^2 - i\,\frac{u_x}{2\,\ell_2}\left(\vec{p}_2 - \ell_2\,\frac{\vec{x}}{u_x}\right)^2 + i\, p_1^i\, \theta_{ij}\, p_2^j}\,, \\ &\sim e^{-i\,\frac{1}{u_x}\,\vec{x}_{nc}^2\,(\ell_1 + \ell_2)}\,. \end{aligned} \tag{A.10}$$

One interesting aspect is that the large momentum behaviour of $H_2(\ell_1, \ell_2, x)$ is independent of the non-commutative parameter θ. Using the expressions for $G_2^{(np)}$ we find at the end of the day,

$$\mathcal{M}_{np}^{(2)} \sim \int_0^\infty d\ell_1\, d\ell_2 \left(e^{i(\ell_1+\ell_2)\frac{x_c^2}{2\,u_x}} - e^{-i(\ell_1+\ell_2)\frac{x_c^2}{2\,u_x}} \right). \tag{A.11}$$

In contrast to the tree-level result, we see that the convergence properties at one-loop (for non-planar contributions) depend only on the light cone defined with respect to the commuting directions in spacetime. Once again going through the possible contour rotations leads us to conclude that the commutator is required in this case to vanish outside the light-wedge as in (3.12).

A.2 Lightlike non-commutativity

Here we choose the non zero components of $\theta^{\mu\nu}$ to be those with $\theta_{vi} = \theta^i \neq 0$, with i denoting the spatial directions in $\mathbf{R}^4 \subset \mathbf{R}^{5,1}$. This in particular implies that the phase factor will be

$$P(\vec{p}_1, \vec{p}_2) = e^{i\,\vec{\theta}\cdot(\ell_1\,\vec{p}_2 + \ell_2\,\vec{p}_1)}\,. \tag{A.12}$$

Hence calculating the kernel $H_2(\ell_1, \ell_2, x)$ reduces to

$$\begin{aligned}
H_2(\ell_1, \ell_2, x) \sim & \int d^4\vec{p}_1\, d^4\vec{p}_2\, e^{-i\,\frac{u_x}{2\,\ell_1}\left(\vec{p}_1 - \ell_1\,\frac{\vec{x}}{u_x}\right)^2 - i\,\frac{u_x}{2\,\ell_2}\left(\vec{p}_2 - \ell_2\,\frac{\vec{x}}{u_x}\right)^2 + i\,\vec{\theta}\cdot(\ell_1\vec{p}_2 + \ell_2\vec{p}_1)}\,, \\
= & \int d^4\vec{p}_1 \exp\Bigg(-i\,\frac{u_x}{2\,\ell_1}\left(\vec{p}_1 - \frac{\ell_1}{u_x}\vec{x} - \frac{\ell_1\,\ell_2}{u_x}\,\vec{\theta}\right)^2 + \\
& \qquad + i\,\frac{u_x}{2\,\ell_1}\left(\frac{\ell_1}{u_x}\,\vec{x} - \frac{\ell_1\,\ell_2}{u_x}\,\vec{\theta}\right)^2 - i\,\frac{\ell_1}{2\,u_x}\vec{x}^2 \Bigg) \\
& \times d^4\vec{p}_2 \exp\Bigg(-i\,\frac{u_x}{2\,\ell_2}\left(\vec{p}_2 - \frac{\ell_2}{u_x}\vec{x} - \frac{\ell_1\,\ell_2}{u_x}\,\vec{\theta}\right)^2 + \\
& \qquad + i\,\frac{u_x}{2\,\ell_2}\left(\frac{\ell_2}{u_x}\,\vec{x} - \frac{\ell_1\,\ell_2}{u_x}\,\vec{\theta}\right)^2 - i\,\frac{\ell_2}{2\,u_x}\vec{x}^2 \Bigg)\,,
\end{aligned} \tag{A.13}$$

The integrals over $d^4\vec{p}_{1,2}$ are easily done, but unlike the case of spacelike non-commutativity discussed previously we find that the dominant term for large ℓ_i depends explicitly on θ. In fact, we have

$$G_2^{(np)}(\ell_1, \ell_2, x) \sim \int_0^\infty d\ell_1\, d\ell_2 \exp\left(i\frac{\vec{\theta}\cdot\vec{\theta}}{2\,u_x}\,\ell_1\,\ell_2\,(\ell_1+\ell_2) - i\,\frac{m^2\,u_x}{2}\left(\frac{1}{\ell_1} + \frac{1}{\ell_2}\right)\right). \tag{A.14}$$

In writing the above we have retained only the dominant contribution at large momentum (for it is the part that determines the convergence properties). If we consider the contour rotations for the $\ell_{1,2}$ integrals, continuing $\ell_{1,2} \to i\,\ell_{1,2}$ leaves us with a divergent integral (assuming without loss of generality $u_x > 0$) for $G_2^{(np)}(\ell_1, \ell_2, x)$. One might wonder about continuing into the lower half-plane i.e., $\ell_{1,2} \to -i\,\ell_{1,2}$; this doesn't help because the integral then diverges at small ℓ_i. Hence there is no contour rotation that will allow us to obtain a finite answer, so we are led to conclude that the correct micro-causality condition is as given in (6.8).

B. Non-distinguishing property the spacetime

In this appendix, we demonstrate directly that the spacetime (4.8),

$$ds^2 = r^2 \left(-2\, du\, dv - \beta^2\, r^4\, du^2 + dx_1^2 + dx_2^2 + \frac{dr^2}{r^4} + \frac{1}{r^2}\, d\Omega_5^2 \right), \tag{B.1}$$

is non-distinguishing. In particular, we exhibit a causal curve connecting a point $P = (u_0, v_0, r_0, x_0, \Omega_0)$ in the spacetime to any point $Q = (u_0 + \varepsilon, v_1, r_1, x_1, \Omega_1)$, with $\varepsilon > 0$ arbitrarily small, and arbitrary values of $(v_1, r_1, x_1, \Omega_1)$, where x and Ω capture the 2 transverse directions (x_1, x_2) and the 5 angular directions of the $\mathbf{S}^5$, respectively.

Since along any future directed causal curve the coordinate u must increase, we can parameterise our curve by u, so that

$$\gamma(u) = (u, v(u), r(u), x(u), \Omega(u))\,. \tag{B.2}$$

Then the causality condition implies that for all u,

$$2\,\dot{v} + \beta^2\, r^4 \geq \dot{x}^2 + \frac{\dot{r}^2}{r^4} + \frac{\dot{\Omega}^2}{r^2} \tag{B.3}$$

where we define $\dot{} \equiv \frac{d}{du}$. For simplicity, we break up the curve into five components; the first and last being null curves in the (r, v) plane, while the middle three move in the x, Ω, and v directions, respectively. Letting each component take equal u interval, these curves, γ_i with $i = 1, \ldots, 5$, join the points

$$\begin{aligned}
P &= (u_0, v_0, r_0, x_0, \Omega_0) \\
p_1 &= (u_0 + \frac{\varepsilon}{5}, v_2, R, x_0, \Omega_0) \\
p_2 &= (u_0 + \frac{2\,\varepsilon}{5}, v_2, R, x_1, \Omega_0) \\
p_3 &= (u_0 + \frac{3\,\varepsilon}{5}, v_2, R, x_1, \Omega_1) \\
p_4 &= (u_0 + \frac{4\,\varepsilon}{5}, v_3, R, x_1, \Omega_1) \\
Q &= (u_0 + \varepsilon, v_1, r_1, x_1, \Omega_1)
\end{aligned} \tag{B.4}$$

with the quantities v_2, v_3 and R as defined below. Choosing this joining to be linear, we can write the component curves explicitly as

$$\begin{aligned}
\gamma_1(u) &= (u, v_a(u), r_a(u), x_0, \Omega_0) \\
\gamma_2(u) &= \left(u, v_2, R, \frac{x_1 - x_0}{\varepsilon/5}\, u + \frac{x_0 \left(u_0 + \frac{2\varepsilon}{5}\right) - x_1 \left(u_0 + \frac{\varepsilon}{5}\right)}{\varepsilon/5}, \Omega_0 \right) \\
\gamma_3(u) &= \left(u, v_2, R, x_1, \frac{\Omega_1 - \Omega_0}{\varepsilon/5}\, u + \frac{\Omega_0 \left(u_0 + \frac{3\varepsilon}{5}\right) - \Omega_1 \left(u_0 + \frac{2\varepsilon}{5}\right)}{\varepsilon/5} \right) \\
\gamma_4(u) &= \left(u, \frac{v_3 - v_2}{\varepsilon/5}\, u + \frac{v_2 \left(u_0 + \frac{4\varepsilon}{5}\right) - v_3 \left(u_0 + \frac{3\varepsilon}{5}\right)}{\varepsilon/5}, R, x_1, \Omega_1 \right) \\
\gamma_5(u) &= (u, v_b(u), r_b(u), x_1, \Omega_1)
\end{aligned} \tag{B.5}$$

with

$$\begin{aligned}
r_a(u) &\equiv \rho_a\, u + \mu_a \\
v_a(u) &\equiv v_0 + \frac{\rho_a}{6}\left(\frac{1}{r_0^3} - \frac{1}{r_a(u)^3}\right) - \frac{\beta^2}{10\rho_a}\left(r_a(u)^5 - r_0^5\right) \\
r_b(u) &\equiv \rho_b\, u + \mu_b \\
v_b(u) &\equiv v_3 + \frac{\rho_b}{6}\left(\frac{1}{R^3} - \frac{1}{r_b(u)^3}\right) - \frac{\beta^2}{10\rho_b}\left(r_b(u)^5 - R^5\right),
\end{aligned} \tag{B.6}$$

where we define the constants

$$\begin{aligned}
\rho_a &= \frac{R - r_0}{\varepsilon/5}, \qquad \mu_a = \frac{r_0\left(u_0 + \frac{\varepsilon}{5}\right) - R\, u_0}{\varepsilon/5} \\
\rho_b &= -\frac{R - r_1}{\varepsilon/5}, \qquad \mu_b = \frac{R\,(u_0 + \varepsilon) - r_1\,\left(u_0 + \frac{4\varepsilon}{5}\right)}{\varepsilon/5} \\
v_2 &= v_0 + \frac{\rho_a}{6}\left(\frac{1}{r_0^3} - \frac{1}{R^3}\right) - \frac{\beta^2}{10\rho_a}\left(R^5 - r_0^5\right) \\
v_3 &= v_1 + \frac{\rho_b}{6}\left(\frac{1}{r_1^3} - \frac{1}{R^3}\right) - \frac{\beta^2}{10\rho_b}\left(R^5 - r_1^5\right).
\end{aligned} \tag{B.7}$$

These ensure that the curve is connected, joining the points (B.4), and (B.6) ensure that γ_1 and γ_5 are null. Finally, to guarantee causality of γ_2, γ_3 and γ_4, we must choose R bounded from below by

$$R \geq \max\left\{\left(\frac{x_1 - x_0}{\beta\,\varepsilon/5}\right)^{1/2}, \left(\frac{\Omega_1 - \Omega_0}{\beta\,\varepsilon/5}\right)^{1/3}, \left(\frac{v_2 - v_3}{\beta^2\,\varepsilon/10}\right)^{1/4}\right\}. \tag{B.8}$$

Physically, we want to take advantage of the large-r region, where the $-\beta^2\, r^4\, du^2$ term in the metric (4.8)C allows causality for large changes in the other coordinates. A schematic representation of such a causal curve in $\{u, v, r\}$ space is sketched in figure 1. Note that this is only possible for nonzero β.

Hence, a crucial feature of our construction of the causal curve γ is that it involves arbitrarily large radii, scaling as inverse positive power of $\Delta u = \varepsilon$. In other words, it is exactly due to the asymptotic region that the spacetime is non-distinguishing. As commented above, this bears two important consequences: it lets us extract this causal property directly from the dual boundary theory, and it pacifies possible causality-violating quantum fluctuations.

Likewise, in the tamer case of the spacetime (2.1), the holographic dual to the $\mathcal{N} = 4$ SYM with spacelike non-commutativity, it is important that the light wedge structure appears only asymptotically. For otherwise, if the spacetime exhibited a light-wedge structure even in the interior, then any two points separated only in the non-commutative directions (along the wedge) would have identical past/future sets — the spacetime would be non-distinguishing. But we know that this is not the case, as (2.1) admits a time function: it is stably causal and therefore distinguishing.

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C. Other solutions generated by null Melvin twist

In this appendix we write down metrics generated by applying the null Melvin twist transformations on other geometries.

C.1 Twisting along angular isometries

Instead of performing the null Melvin twist on the planar directions (x_1, x_2) we could instead have used some angular directions. The most general such configuration can be generated by starting with the D3-brane solution written as (4.3), with

$$\begin{aligned} dx_1^2 + dx_2^2 &= d\rho^2 + \rho^2\, d\phi_1^2 \\ d\Omega_5^2 &= d\chi^2 + \cos^2\chi\, d\phi_2^2 + \frac{1}{4}\sin^2\chi\, \left(d\theta^2 + d\phi_3^2 + d\psi^2 + 2\cos\theta\, d\phi_3\, d\psi\right). \end{aligned} \tag{C.1}$$

We can in this case perform the null Melvin twist by

$$d\phi_i \;\to\; d\phi_i + dy\,, \qquad \text{for } i = 1, 2, 3\,. \tag{C.2}$$

This leads to the metric

$$\begin{aligned} ds^2 &= -H^{-\frac{3}{2}}\,\beta^2\left(\rho^2 + r^2\, H\right)(dt+dy)^2 + \frac{1}{\sqrt{H}}\left(-dt^2 + dy^2 + d\rho^2 + \rho^2\, d\phi_1^2\right) \\ &\quad + \sqrt{H}\left(dr^2 + r^2\left[d\chi^2 + \cos^2\chi\, d\phi_2^2 + \frac{1}{4}\sin^2\chi\,\left(d\theta^2 + d\phi_3^2 + d\psi^2 + 2\cos\theta\, d\phi_3\, d\psi\right)\right]\right) \\ B &= \beta\,(dt+dy)\wedge\left(\frac{\rho^2}{H}\, d\phi_1 + r^2\cos^2\chi\, d\phi_2 + \frac{r^2}{4}\sin^2\chi\,(d\phi_3 + \cos\theta\, d\psi)\right) \end{aligned} \tag{C.3}$$

While the metric (C.3) is interesting in its own right, we will mostly focus on the simpler case presented in (4.6).

C.2 Non-extremal null Melvin twisted D3-brane geometry

To obtain the supergravity dual to thermal lightlike non-commutative $\mathcal{N} = 4$ SYM, we need to start with a non-extremal D3-brane geometry and carry out the null Melvin twist duality transformation. Recall that the metric for the non-extremal D3-brane takes the form

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-f(r)\, dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + \sqrt{H(r)}\left(\frac{dr^2}{f(r)} + r^2\, d\Omega_5^2\right), \tag{C.4}$$

with

$$f(r) = 1 - \frac{r_0^4}{r^4}\,. \tag{C.5}$$

In this case the non-extremality of the solution implies that we no longer have the SO(1,1) symmetry along the D3-brane world-volume. However, $\left(\frac{\partial}{\partial y}\right)^a$ is still a good spacelike Killing vector and we have the requisite symmetries to proceed with the duality chain.

Carrying out Steps **1** through **7** outlined in section 4.1, we obtain the metric for the null Melvin twisted non-extremal D3-brane:

$$ds^2 = \frac{H\left(-f dt^2 + dy^2 + dx_1^2 + dx_2^2\right) - \beta^2 f\,(dt+dy)^2 + \frac{1}{2}\beta^2\,(1-f)(dx_1 - dx_2)^2}{\sqrt{H}\,\left(H + \beta^2\,(1-f)\right)} + \\ + \sqrt{H}\left(\frac{dr^2}{f(r)} + r^2\,d\Omega_5^2\right). \qquad \text{(C.6)}$$

Here as in (4.6), we have twisted along the non-compact directions (x_1, x_2) for simplicity. We have also refrained from writing down the explicit expressions for the p-form fields supporting the solution. This solution was also discussed in [38].

The metric is a bit more complicated than (4.6), but it is easy to check that in the decoupling limit (replacing $H(r) \to R^4/r^4$) the asymptotic form of the metric coincides with that of (4.8). Since it is the asymptotic region that is responsible for the non-distinguishing character of the spacetime, we conclude that (C.6) is also a non-distinguishing spacetime. That is, since the decoupling limit of (C.6) and (4.8) have a similar large r behaviour, two points which are sufficiently far away from the black hole ought to have the same causal future/past, implying that the spacetime is non-distinguishing.

C.3 Null Melvin twist of Klebanov-Strassler

The Klebanov-Strassler solution is given by the metric

$$ds^2 = h(r)^{-1/2}\left(-dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + h(r)^{1/2}\,ds_6^2\,, \qquad \text{(C.7)}$$

where ds_6^2 denotes the metric of the deformed conifold and r is a radial coordinate that measures the size of the base. The solution for the warp factor $h(r)$ is given as an integral expression

$$h(r) = C\int_r^\infty dz\,\frac{z\coth z - 1}{\sinh^2 z}\left(\sinh(2z) - 2\,z\right)^{\frac{1}{3}}\,, \qquad \text{(C.8)}$$

and the metric is supported by both five-form and three-form fluxes. Asymptotically, this metric approaches the Klebanov-Tseytlin (KT) geometry [39] with

$$h(r) = \frac{R^4}{r^4}\,\log\left(\frac{r}{r_s}\right), \qquad \text{(C.9)}$$

where R is the length scale set by the number of fractional branes and r_s the IR scale where we should revert to the original solution (C.8).

For our purposes it will suffice to note that we have a flat metric on $\mathbf{R}^{3,1}$ which is non-trivially warped. Thus we can carry out the null Melvin twist along the translational isometries. Carrying out the steps in the duality chain we will end up with a metric that is analogous to (4.8). Apart from the fact that we have to replace the $\mathbf{S}^5$ by the base of the deformed conifold we have no new change. In the large r limit, where we have the KT solution, the base is the Einstein space $T^{1,1}$. The astute reader will have realized that we have already taken the near horizon limit in writing the metric (C.7). So the spacetime dual to lightlike non-commutative deformed $\mathcal{N} = 1$ cascade theory is

$$ds^2 = -\,\beta^2\,h(r)^{-\frac{3}{2}}\,(dt+dy)^2 + \frac{1}{\sqrt{h(r)}}\left(-dt^2 + dy^2 + dx_1^2 + dx_2^2\right) + \sqrt{h(r)}\,ds_6^2\,, \qquad \text{(C.10)}$$

which asymptotically looks like

$$ds^2 = -\frac{r^2}{R^2\sqrt{\log(r/r_s)}}\Bigg(-2\,du\,dv + dx_1^2 + dx_2^2 - \beta^2\,\frac{r^4}{R^4\,\log(r/r_s)}\,du^2 + \\ + \frac{R^4}{r^4}\,\log\left(\frac{r}{r_s}\right)\left(dr^2 + r^2\,d\Omega_5^2\right)\Bigg). \tag{C.11}$$

The only difference from the non-distinguishing spacetime (4.8) is the logarithmic pieces. However, it is easy to check that these do not affect the causal structure of the spacetime in an essential manner.

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